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Stabilization and Controllability of a Class of Nonholonomic Systems *

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Abstract: In this note, the stabilization and controllability of a class of nonholonomic control systems are considered. First of all, it is shown that even though in general a smooth state feedback control, which stabilizes the system, does exist[1], the existence depends on the initial position of the system. It does exist except a very limited measure-zero set. Then we show that the systems are globally controllable by piecewise smooth controls. As a corollary, the systems are finite time stabilizable by piecewise smooth control. The proofs are constructive so the controls are provided precisely.

Key words: nonholonomic system; stabilization; controllability; state feedback; finite time control

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一类非完整约束系统的镇定与能控性

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摘要: 考虑非完整约束系统的镇定与能控性问题.文中首先证明了虽然一般地说这类系统不能由光滑反馈镇定,但它的存在性依赖于初值.当初值在一个零测集外时,这种反馈镇定确实存在.然后我们证明了这类系统在分段光滑控制下全局可控.由于证明是构造性的,它给出了相应的控制.

关键词: 非完整约束系统; 镇定; 能控性; 状态反馈; 有限时间控制

In the last decade, nonholonomic control systems received a lot of attention. Because many practical engineering problems, such as moving robot, space craft etc., can be described as this kind of control systems[2]. Since in general a nonholonomic control system can not be stabilized by smooth state feedback control, non-smooth and discontinuous controls are investigated[3~5].

The following nonholonomic system is the well-known Brockett integrator. It was shown in [1] that the system is locally controllable and it can not be stabilized by smooth state feedback control.

$$\begin{cases} \dot{x}_1 = u_1, \\ \dot{x}_2 = u_2, \\ \dot{x}_3 = x_2 u_1 - x_1 u_2. \end{cases} \tag{1}$$

This system has been widely discussed. Many different control techniques have been used to stabilize it. A recent research is done by using non-regular linearization technique[6].

This note considers a class of nonholonomic systems described as

$$\begin{cases} \dot{x} = u, \ x \in \mathbb{R}^n, \ u \in \mathbb{R}^n, \\ \dot{z} = x^{\mathrm{T}} J u, \end{cases} \tag{2}$$

where J is a skew-symmetric and invertible matrix. For J to be nonsingular n should be even. Note that system (1) is a particular case of (2).

1 Feedback stabilization

To begin with, we propose a quadratic Lyapunov function as

$$H(x,z) = \frac{1}{2}(x^{\mathrm{T}}x + \phi(z)^2),$$

where $\phi(z)$ is smooth and strictly monotone with $\phi(0) = 0$. Then

$$\dot{H} = x^{\mathrm{T}}u + \phi(z)\phi'(z)x^{\mathrm{T}}Ju = x^{\mathrm{T}}(I + \phi(z)\phi'(z)J)u. \tag{3}$$

If we assume

A1) $J^2 = -I$.

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Note that system (1) satisfies this assumption. Then it is very natural to choose a control as

$$u = -(I - \phi(z)\phi'(z)J)x. \tag{4}$$

Then

$$\dot{H} = -[1 + (\phi(z)\phi'(z))^2]\|x\|^2 \leqslant 0,$$

which implies the following

Proposition 1 Assume A1). Then there exists a smooth state feedback such that the closed-loop system of (2) is globally stable.

Proposition 1 is not very interesting because we are more interested in the asymptotical stability. Moreover, the assumption A1) is too restrictive. But motivated by the control form (4), we may propose the following control

$$u = -(I + \xi(z)J^{-1})x, \tag{5}$$

where $\xi(z)$ is a smooth function of z. Using this control we can find the following result, which shows that in most cases (precisely, for the initial state on a generic subset of the state space, $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,z)\}$) system (2) can still be stabilized by smooth state feedback.

Theorem 1 If the initial condition for x is not zero, i.e., $x(0) \neq 0$, system (2) can be stabilized by smooth state feedback control.

Proof Using (5), it is seen that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x\|^2 = -2x^{\mathrm{T}}(I + \xi(z)J^{-1})x = -2\|x\|^2.$$

Denote $X_0 = \|x_0\|^2$, then

$$\|x\|^2 = X_0 \mathrm{e}^{-2t}. \tag{6}$$

Now we have

$$\dot{z} = -x^{\mathrm{T}}Jx - x^{\mathrm{T}}\xi(z)Ix = -\xi(z)\|x\|^2. \tag{7}$$

It follows from (6),(7) that

$$\dot{z} = -\xi(z)X_0\mathrm{e}^{-2t},$$

or

$$\int \frac{\mathrm{d}z}{\xi(z)} = -\int X_0 \mathrm{e}^{-2t}\mathrm{d}t.$$

Denote $\Phi(z) = \int \frac{\mathrm{d}z}{\xi(z)}$. Then we have

$$\Phi(z) - \Phi(z_0) = \frac{X_0}{2}(\mathrm{e}^{-2t} - 1). \tag{8}$$

It follows that

$$\lim_{t\to\infty}\Phi(z) = \Phi(z_0) - \frac{X_0}{2}. \tag{9}$$

Now it is clear that if $\Phi(z)$ satisfies the following C1) ~ C3), the closed-loop system is asymptotically stable, i.e., the system (2) can be stabilized by control (5).

C1) $\Phi(0) = 0$;

C2) $\Phi(z) \to 0$ implies $z \to 0$;

C3) $$\Phi(z_0) - \frac{X_0}{2} = 0. \tag{10}$$

A feasible function, satisfying C1) ~ C3), is

$$\Phi(z) = \mu\ln(1+|z|), \text{ with } \mu = \frac{X_0}{2\ln(1+|z_0|)}, \quad z_0 \neq 0. \tag{11}$$

Then

$$\frac{1}{\xi(z)} = \frac{\mathrm{d}}{\mathrm{d}z}\Phi(z) = \begin{cases} \dfrac{\mu}{1+z}, & z \geqslant 0, \\ \dfrac{\mu}{z-1}, & z < 0. \end{cases}$$

It follows that

$$\xi(z) = \begin{cases} \dfrac{2(1+z)\ln(1+|z_0|)}{X_0}, & z \geqslant 0, \\ \dfrac{2(z-1)\ln(1+|z_0|)}{X_0}, & z < 0. \end{cases} \tag{12}$$

It is easy to find that from (5) the constraint $z_0 \neq 0$ can be removed. In fact, $z_0 = 0$ implies $\xi(z) = 0$. Then the corresponding control (5) does stabilize the system.

To know that control (5) with (12) is a smooth control we have only to show that along each trajectory of the closed-loop system (4) will take only one branch. This fact derives from the following claim: If $z_0 \geqslant 0$ then $z(t) \geqslant 0$, $\forall t \geqslant 0$, and If $z_0 < 0$ then $z(t) < 0$, $\forall t > 0$.

If $z_0 = 0$, then $\xi(z_0) = 0$ and $z(t) = 0$, $\forall z$.

If $z_0 > 0$, then $\xi(z_0) > 0$, z is decreasing. Since $\lim_{t\to\infty} z = 0$, and $\dot{z} < 0$, then $z(t) > 0$, $\forall t$.

If $z_0 < 0$, then $\xi(z_0) < 0$, z is increasing. A similar argument shows that $z(t) < 0$, $\forall t$.

From (9), the stability is not robust. It depends on the initial measurement Z_0 and X_0. Intuitively, we may replace Z_0 and X_0 by Z and X respectively. It simply means we adjust the initial condition from time to time. It is indeed true.

Proposition 2 If the initial condition $x(0) \neq 0$, system (2) can be stabilized by smooth state feedback control (5) with

$$\xi(z) = \begin{cases} \dfrac{2(1+z)\ln(1+|z|)}{\|x\|^2}, & z > 0, \\ \dfrac{2(z-1)\ln(1+|z|)}{\|x\|^2}, & z < 0, \\ 0, & z = 0. \end{cases} \tag{13}$$

Proof Since $\| x \|^2 > 0, \forall t$, similar to the proof of Theorem 1 we can show that the control is smooth because $z_0 > 0$, $z_0 = 0$ and $z_0 < 0$ imply $z(t) > 0, \forall t$, $z(t) = 0, \forall t$ and $z(t) < 0, \forall t$ respectively. We assume $z_0 > 0$($z_0 = 0$ is obvious, and $z_0 < 0$ can be proved similarly). Then from (7)

$$\dot{z} = -\frac{2(1+z)\ln(1+z)}{x^{\mathrm{T}}x} \| x \|^2 = -2(1+z)\ln(1+z).$$

The solution for this ODE is $\ln(1+z) = \ln(1+z_0)\mathrm{e}^{-2t}$.

It is ready to see that $\lim_{t\to\infty} z(t) = 0$.

For the stabilization of system (2) with non-smooth control in the case $x_0 = 0$, it will be solved as a byproduct of the investigation of the controllability.

2 Controllability

It is known that system (1) is locally controllable[4]. In this section we will show that (2) is globally controllable in a very strong sense: a system is called globally controllable if given any starting position x_0 and destination x_e there exist piece-wise smooth state-feedback controls, such that the trajectory of the closed-loop system satisfies $x(0) = x_0$ and $x(T) = x_e$ for some $T > 0$.

We will show that system (2) is globally controllable by constructing the required controls.

Theorem 2 System (2) is globally controllable by piece-wise smooth state feedback control.

Proof Given any $(x_0, z_0), (x_e, z_e) \in \mathbb{R}^{n+1}$, we will construct controls, which drives the system from (x_0, z_0) to (x_e, z_e).

Case 1 $x_e \neq 0$. The controls will be designed in 4 steps.

Step 1 Construct u_1 as

$$u_1 = \sqrt{2}x_e - x_0, \quad 0 \leqslant t \leqslant 1. \tag{14}$$

Then $x(t) = x_0 + (\sqrt{2}x_e - x_0)t, x_1 := x(1) = \sqrt{2}x_e$.

In the following, for notational ease we will use x_t for $x(t)$, etc.

$$\dot{z} = [x_0 + (\sqrt{2}x_e - x_0)t]^{\mathrm{T}} J(\sqrt{2}x_e - x_0) = \sqrt{2}x_0 J x_e.$$

Then $z(t) = \sqrt{2}x_0 J x_e t + z_0, z_1 = \sqrt{2}x_0{}^{\mathrm{T}} J x_e + z_0$.

Step 2 Construct u_2 as

$$u_2 = \frac{z_e - z_1}{2\| x_e \|^2} J^{-1} x := \mu_1 J^{-1} x, \quad 1 < t \leqslant 3. \tag{15}$$

Then

$$\dot{x} = \mu_1 J^{-1} x. \tag{16}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \| x \|^2 = 2x^{\mathrm{T}} \mu_1 J^{-1} x = 0.$$

Using this fact, we have

$$\dot{z} = x^{\mathrm{T}} \frac{z_e - z_1}{2\| x_e \|^2} x = z_e - z_0.$$

Then $z(t) = z_1 + (z_e - z_1)(t-1)$, and $z_3 = z_1 + 2(z_e - z_1)$.

According to (16), we have

$$x(t) = \mathrm{e}^{\mu_1 J^{-1}(t-1)} x_1 \text{ and } x_3 = \sqrt{2}\mathrm{e}^{2\mu_1 J^{-1}} x_e. \tag{17}$$

Step 3 Construct u_3 as

$$u_3 = (\frac{\sqrt{2}}{2} - 1)x_3, \quad 3 < t \leqslant 4. \tag{18}$$

Then

$$\dot{x} = (\frac{\sqrt{2}}{2} - 1)x_3.$$

Hence

$$x(t) = x_3 + (\frac{\sqrt{2}}{2} - 1)x_3(t-3) \text{ and } x_4 = \frac{\sqrt{2}}{2}x_3 = \mathrm{e}^{2\mu_1 J^{-1}} x_e. \tag{19}$$

Using (19), we have

$$\dot{z} = [1 + (\frac{\sqrt{2}}{2} - 1)(t-3)] x_3^{\mathrm{T}} J x_3 = 0.$$

Then $z_4 = z_3 = z_1 + 2(z_e - z_1)$.

Step 4 Construct u_4 as

$$u_4 = -\frac{z_e - z_1}{\| x_e \|^2} J^{-1} x := -\mu_2 J^{-1} x, \quad 4 < t \leqslant 5. \tag{20}$$

Note that $\mu_2 = 2\mu_1$. Then

$$\dot{x} = -\mu_2 J^{-1} x. \tag{21}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \| x \|^2 = -2x^{\mathrm{T}} \mu_2 J^{-1} x = 0.$$

Using this fact, we have

$$\dot{z} = -x^{\mathrm{T}} \frac{z_e - z_1}{\| x_e \|^2} x = -(z_e - z_1).$$

Then $z(t) = z_4 - (z_e - x_1)(t-4)$, and $z_5 = z_1 + 2(z_e - z_1) - (z_e - x_1) = z_e$.

According to (21), we have

$$x(t) = \mathrm{e}^{-\mu_2 J^{-1}(t-4)} x_4 \text{ and } x_5 = \mathrm{e}^{-\mu_2 J^{-1}} \mathrm{e}^{2\mu_1 J^{-1}} x_e = x_e. \tag{22}$$

We reach the destination.

Case 2 $x_e = 0$.

Choose an auxiliary $\tilde{x}_e \neq 0$ to replace x_e. Using Step 1 to Step 4 we have: $x_5 = \tilde{x}_e$ and $z_5 = z_e$. Then we set

$$u_5 = -\tilde{x}_e. \tag{23}$$

Based on the above analysis, we can easily find that $x_6 = 0 = x_e$ and $z_6 = z_5 = z_e$.

Remark 1 In the above case 2, if $x_0 \neq 0$ we can simply choose the following controls to reach the destination:

Step 1 Choose

$$u_1 = \frac{z_e - z_0}{\| x_0 \|^2} J^{-1} x := \mu x,\ 0 \leqslant t \leqslant 1. \tag{24}$$

Then $x_1 = e^{\mu J^{-1}} x_0$ and $z_1 = z_e$.

Step 2 Choose

$$u_2 = -x_1,\ 1 < t \leqslant 2. \tag{25}$$

Then $x_2 = 0 = x_e$ and $z_1 = z_1 = z_e$.

Setting $x_e = 0$ and $z_e = 0$, we have

Corollary 1 System (2) is finite time stabilizable by piece-wise smooth state-feedback controls.

Remark 2 Combining the techniques used in this section and the last section we may also stabilize system (2) for $x_0 = 0$ in two steps. Fist we use constant control

$$u_1 = c \neq 0. \tag{26}$$

Then $x(t) = ct$ and it is easy to see that $z(t) \equiv z_0$. In next step we use u_2 as (5) with $\xi(z)$ as in (12) or (13). But the overall control can never be even continuous. For example, we can find some time t such that $u_1 = u_2$. i.e.,

$$-(I + \xi(z_0)J^{-1})tc = c,$$

then $-1/t$ is an eigenvalue of $(I + \xi(z_0)J^{-1})$. But it is easy to see that this matrix can not have a real eigenvalue, which leads to a contradiction.

Remark 3 In fact the condition J is invertible in this note can be removed. As long as $J \neq 0$ we can always find a nonsingular matrix M such that after a congruent transformation we have

$$M^{\mathrm{T}} J M = \begin{pmatrix} J_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where J_0 is a $k \times k$ nonsingular matrix for some $k < n$.

Then we may make a coordinate transformation $x = My$ and set the control as $u = Mv$. Let $y = (y^1, y^2)$ with $y^1 \in \mathbb{R}^k$. Then system (2) becomes

$$\begin{cases} \dot{y}^1 = v^1,\ \dot{y}^2 = v^2, \\ \dot{z} = (y^1)^{\mathrm{T}} J_0 v^1. \end{cases} \tag{27}$$

Since the decoupled sub-system $\dot{y}^2 = v^2$ is trivially controllable, all the conclusions in the note still hold true by applying the techniques to the sub-system

$$\begin{cases} \dot{y}^1 = v^1, \\ \dot{z} = (y^1)^{\mathrm{T}} J_0 v^1. \end{cases} \tag{28}$$

3 Conclusion

In this paper a class of nonholonomic control systems were considered. The systems described by (2) are a generalization of the famous example (1), introduced in [1]. Even though in general system (1) is not smooth state feedback stabilizable, we proved that as $x_0 \neq 0$ there exists a smooth state feedback control, which stabilized the system. Then the system (2) was shown to be globally controllable by piecewise smooth control. A universal control law was proposed to drive the system from any initial position to any destination. As a corollary, it was also shown that the system (2) is finite time stabilizable by the piecewise smooth control, which is better than the exponential stabilization.

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