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广义非保守系统的新型最小作用量原理*

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摘要 分别建立了广义非保守系统的 Hamilton - Tabarrok - Leech 正则方程和 Raitzin - Tabarrok - Leech 正则方程. 给出了广义非保守系统的三种新型最小作用量原理: Lagrange - Tabarrok - Leech 最小作用量原理, Raitzin - Tabarrok - Leech 最小作用量原理和 Lagrange - Raitzin - Tabarrok - Leech 最小作用量原理, 并举例说明这些原理的应用.

关键词 广义经典力学,非保守系统,最小作用量原理

引 言

1744 年法国学者 P. L. Maupertuis 最先提出了完整保守系统最小作用量原理,到 1760 年由 Lagrange 给以明确论证:后人称为 Lagrange 最小作用量原理[1]. 它表明对系统的真实运动来说, Lagrange 作用量的全变分为零.

1985 年,Mei 研究了非完整保守系统在广义坐标和准坐标下的 Lagrange 最小作用量原理[2],1990 年他又将该原理推广到变质量非完整非保守系统,给出原理的 Hölder 形式和 СусЛоВ 形式[3].

1991 年 Qiao 给出了非完整非保守系统的最小作用量原理[4] 及广义力学系统的最小作用量原理[5].

2002 年,由 Tabarrok 和 Leech 研究了具有二阶导数的泛函的 Hamilton 力学[6]. 沿用普通分析力学中的研究思路,引入两个新广义动量和新 Hamilton 函数,将保守系统的四阶 Euler - Lagrange 方程化为四个一阶的运动方程,并利用它建立了泛函依赖二阶导数的新型最小作用量原理.

本文推广了 T - L 的工作,在力学系统不存在能量积分的条件下,给出用新变量表示的三种广义非保守系统最小作用量原理,所得结果更具有一般性.

1 非保守系统的 Hamilton-Tabarrok-Leech 正则方程

考虑非势力作用的系统,其位形由 n 个广义坐标 $q_1, q_2, \cdots, q_n$ 确定.

Lagrange 函数为

$$L = L(t, q^j, q^j_{(1)}, q^j_{(2)}) \tag{1}$$

非势广义力为

$$Q^j = Q^j(t, q^j, q^j_{(1)}, q^j_{(2)}) \tag{2}$$

系统的 Lagrange 方程为

$$\frac{\partial L}{\partial q^j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q^j_{(1)}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial q^j_{(2)}}\right) = -Q^j$$
$$(j = 1, 2, \cdots, n) \tag{3}$$

其中

$$q^j_{(1)} = \dot{q}^j = \frac{\mathrm{d}q^j}{\mathrm{d}t}, \quad q^j_{(2)} = \frac{\mathrm{d}}{\mathrm{d}t}q^j_{(1)} = \frac{\mathrm{d}^2 q^j}{\mathrm{d}t^2} \tag{4}$$

文中其余各变量的导数表示记号,仿式(4)写出.

定义广义动量为

$$v^j = \frac{\partial L}{\partial q^j_{(2)}}, \quad p^j = \frac{\partial L}{\partial q^j_{(1)}} - v^j_{(1)} \tag{5}$$

于是,方程(3)可写为

$$\frac{\partial L}{\partial q^j} = \frac{\mathrm{d}p^j}{\mathrm{d}t} - Q^j = p^j_{(1)} - Q^j \tag{6}$$

引入 Hamilton 函数为

$$H(t, q^j, q^j_{(1)}, p^j, v^j_{(1)}) = p^j q^j_{(1)} + v^j q^j_{(2)} - L \tag{7}$$

则非保守系统的 H-T-L 正则方程为

$$\frac{\mathrm{d}p^j}{\mathrm{d}t} = -\frac{\partial H}{\partial q^j} + Q^j, \quad \frac{\mathrm{d}q^j}{\mathrm{d}t} = \frac{\partial H}{\partial p^j}$$
$$\frac{\mathrm{d}v^j}{\mathrm{d}t} = -\frac{\partial H}{\partial q^j_{(1)}}, \quad \frac{\mathrm{d}q^j_{(1)}}{\mathrm{d}t} = \frac{\partial H}{\partial v^j} \tag{8}$$

2 非保守系统的 Lagrange-Tabarrok-Leech 最小作用量原理

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假设非保守系统不存在能量积分,即

$$H = p^j q^j_{(1)} + v^j q^j_{(2)} - L = h \neq \mathrm{const} \tag{9}$$

系统的 Hamilton 作用量为

$$W = \int_{t_1}^{t_2} L(t, q^j, q^j_{(1)}, q^j_{(2)})\,\mathrm{d}t \tag{10}$$

根据全变分与等时变分之间的关系

$$\triangle W = \delta W + \dot{W}\Delta t$$

于是,有

$$\triangle W = \triangle \int_{t_1}^{t_2} L\mathrm{d}t = \int_{t_1}^{t_2} \delta L\mathrm{d}t + (L\Delta t)\mid_{t_1}^{t_2} = \int_{t_1}^{t_2} (\frac{\partial L}{\partial q^j}\delta q^j + \frac{\partial L}{\partial q^j_{(1)}}\delta q^j_{(1)} + \frac{\partial L}{\partial q^j_{(2)}}\delta q^j_{(2)})\,\mathrm{d}t + (L\Delta t)\mid_{t_1}^{t_2} \tag{11}$$

利用等时变分条件 $d(\delta q^j) = \delta(dq^j)$,将式(11)进行分部积分,得

$$\triangle W = \int_{t_1}^{t_2} [\frac{\partial L}{\partial q^j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q^j_{(1)}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\frac{\partial L}{\partial q^j_{(2)}})]\delta q^j \mathrm{d}t + [\frac{\partial L}{\partial q^j_{(1)}}\delta q^j + \frac{\partial L}{\partial q^j_{(2)}}\delta q^j_{(1)} - \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial q^j_{(2)}})\delta q^j + L\Delta t]\mid_{t_1}^{t_2} = \int_{t_1}^{t_2} -Q^j\delta q^j \mathrm{d}t + [\frac{\partial L}{\partial q^j_{(1)}}\delta q^j + \frac{\partial L}{\partial q^j_{(2)}}\delta q^j_{(1)} - \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial q^j_{(2)}})\delta q^j + L\Delta t]\mid_{t_1}^{t_2} \tag{12}$$

注意到 $\Delta q^j_{(m-1)} = \delta q^j_{(m-1)} + q^j_m \Delta t\ (m = 1, 2)$,并考虑式(5)和式(9),有

$$\triangle W = \int_{t_1}^{t_2} -Q^j\delta q^j \mathrm{d}t + [\frac{\partial L}{\partial q^j_{(1)}}(\Delta q^j - q^j_{(1)\Delta t}) + \frac{\partial L}{\partial q^j_{(2)}}(\Delta q^j_{(1)} - q^j_{(2)}\Delta t) - \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial q^j_{(2)}})(\Delta q^j - q^j_{(1)}\Delta t) + L\Delta t]\mid_{t_1}^{t_2} = \int_{t_1}^{t_2} -Q^j\delta q^j \mathrm{d}t + (p^j\Delta q^j + v^j\Delta q^j_{(1)} - h\Delta t)\mid_{t_1}^{t_2} \tag{13}$$

因为 $\Delta(ht) = t\Delta h + h\Delta t$,于是式(13)可写为

$$\Delta(W + ht\mid_{t_1}^{t_2}) = \int_{t_1}^{t_2} -Q^j\delta q^j \mathrm{d}t + (p^j\Delta q^j + v^j\Delta q^j_{(1)} + t\Delta h)\mid_{t_1}^{t_2} \tag{14}$$

又因

$$d(ht) = tdh + hdt \tag{15}$$

将式(15)代入式(14),得

$$\Delta(W + \int_{t_1}^{t_2} t\mathrm{d}h + \int_{t_1}^{t_2} h\mathrm{d}t) = (p^j\Delta q^j + v^j\Delta q^j_{(1)} + t\Delta h)\mid_{t_1}^{t_2} - \int_{t_1}^{t_2} Q^j\delta q^j \mathrm{d}t \tag{16}$$

注意到

$$W + \int_{t_1}^{t_2} h\mathrm{d}t = \int_{t_1}^{t_2} (L + h)\,\mathrm{d}t = \int_{t_1}^{t_2} (p^j q^j_{(1)} + v^j q^j_{(2)})\,\mathrm{d}t \tag{17}$$

$$h = h(t), \quad \mathrm{d}h = h_{(1)}\mathrm{d}t \tag{18}$$

于是,式(16)成为

$$\Delta \int_{t_1}^{t_2} (p^j q^j_{(1)} + v^j q^j_{(2)} + th_{(1)})\,\mathrm{d}t = (p^j\Delta q^j + v^j\Delta q^j_{(1)} + t\Delta h)\mid_{t_1}^{t_2} - \int_{t_1}^{t_2} Q^j\delta q^j \mathrm{d}t \tag{19}$$

假设

$$\Delta h\mid_{t_1} = \Delta h\mid_{t_2} = 0, \quad \Delta q^j\mid_{t_1} = \Delta q^j\mid_{t_2} = 0, \quad \Delta q^j_{(1)}\mid_{t_1} = \Delta q^j_{(2)}\mid_{t_2} = 0 \tag{20}$$

利用式(20),可由式(19),得

$$\Delta \int_{t_1}^{t_2} (p^j q^j_{(1)} + v^j q^j_{(2)} + th_{(1)})\,\mathrm{d}t + \int_{t_1}^{t_2} Q^j\delta q^j \mathrm{d}t = 0 \tag{21}$$

方程(21)就是广义非保守系统的 Lagrange-Tabarrok-Leech 最小作用量原理.

讨论特殊情况:

2.1 若广义非势力 $Q^j = 0$,则式(21)成为

$$\Delta \int_{t_1}^{t_2} (p^j q^j_{(1)} + v^j q^j_{(2)} + th_{(1)})\,\mathrm{d}t = 0 \tag{22}$$

2.2 平稳保守系统,有 $h_{(1)} = 0, Q^j = 0$,于是,得

$$\Delta \int_{t_1}^{t_2} (p^j q^j_{(1)} + v^j q^j_{(2)})\,\mathrm{d}t = 0 \tag{23}$$

方程(23)就是在 2002 年由 Tabarrok-Leech 所得的结果[6].

3 非保守系统的 Raitzin-Tabarrok-Leech 型正则方程

对于非保守系统(3),引入 Raitzin 正则变量

$$s^j = q^j_{(1)}, \quad s^j_{(1)} = q^j_{(2)}, r^j = \frac{\partial L}{\partial q^j}$$
$$u^j = v^j_{(1)} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q^j_{(2)}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial s^j_{(1)}} \tag{24}$$

及 Raitzin 函数为

$$R(t, s^j, s^j_{(1)}, r^j, u^j) = L(t, q^j, s^j, s^j_{(1)}) - r^j q^j - u^j s^j \tag{25}$$

于是,有

$$\frac{\partial R}{\partial r^j} = -q^j, \quad \frac{\partial R}{\partial u^j} = -s^j$$
$$\frac{\partial R}{\partial s^j} = \frac{\partial L}{\partial s^j} - u^j, \quad \frac{\partial R}{\partial s^j_{(1)}} = \frac{\partial L}{\partial s^j_{(1)}} \tag{26}$$

将式(26)对时间 t 求导数,得

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s^j} = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial s^j} - u^j) = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial s^j} - \frac{\mathrm{d}v^j}{\mathrm{d}t}) = \frac{\mathrm{d}p^j}{\mathrm{d}t} = r^j + Q^j$$

即

$$r^j = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s^j} - Q^j, \quad s^j = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r^j}$$

$$s^j_{(1)} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial u^j}, \quad u^j = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s^j_{(1)}} \tag{27}$$

4 非保守系统的 Raitzin-Tabarrok-Leech 型最小作用量原理

考虑非保守系统(3),其 Raitzin 函数为

$$R = R(t, s^j, s^j_{(1)}, r^j, u^j) \tag{28}$$

于是

$$\Delta R = \frac{\partial R}{\partial t}\Delta t + \frac{\partial R}{\partial s^j}\Delta s^j + \frac{\partial R}{\partial s^j_{(1)}}\Delta s^j_{(1)} + \frac{\partial R}{\partial r^j}\Delta r^j + \frac{\partial R}{\partial u^j}\Delta u^j \tag{29}$$

对上述变分等式在有限区间$[t_1, t_2]$上进行积分,得

$$\int_{t_1}^{t_2}\Delta R\mathrm{d}t = \int_{t_1}^{t_2}\{\frac{\partial R}{\partial t}\Delta t + \frac{\partial R}{\partial s^j}\Delta s^j + \frac{\partial R}{\partial s^j_{(1)}}\Delta s^j_{(1)} + \frac{\partial R}{\partial r^j}\Delta r^j + \frac{\partial R}{\partial u^j}\Delta u^j\}\mathrm{d}t \tag{30}$$

由于

$$\Delta s^j = \frac{\mathrm{d}}{\mathrm{d}t}\Delta q^j - s^j\frac{\mathrm{d}}{\mathrm{d}t}\Delta t$$

$$\Delta s^j_{(1)} = \frac{\mathrm{d}}{\mathrm{d}t}q^j_{(1)} - s^j_{(1)}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t \tag{31}$$

$$\int_{t_1}^{t_2}\frac{\partial R}{\partial s^j}\Delta s^j\mathrm{d}t = \int_{t_1}^{t_2}\frac{\partial R}{\partial s^j}[\frac{\mathrm{d}}{\mathrm{d}t}\Delta q^j - s^j\frac{\mathrm{d}}{\mathrm{d}t}\Delta t]\mathrm{d}t = -\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial R}{\partial s^j})\Delta q^j\mathrm{d}t - \int_{t_1}^{t_2}(\frac{\partial R}{\partial s^j}\cdot s^j\cdot\frac{\mathrm{d}}{\mathrm{d}t}\Delta t)\mathrm{d}t + \frac{\partial R}{\partial s^j}\Delta q^j\Big|_{t_1}^{t_2} \tag{32}$$

$$\int_{t_1}^{t_2}\frac{\partial R}{\partial s^j_{(1)}}\Delta s^j_{(1)}\mathrm{d}t = \int_{t_1}^{t_2}\frac{\partial R}{\partial s^j_{(1)}}[\frac{\mathrm{d}}{\mathrm{d}t}\Delta q^j_{(1)} - s^j_{(1)}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t]\mathrm{d}t = -\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial R}{\partial s^j_{(1)}})\Delta q^j_{(1)}\mathrm{d}t - \int_{t_1}^{t_2}(\frac{\partial R}{\partial s^j_{(1)}}\cdot s^j_{(1)}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\Delta t)\mathrm{d}t + \frac{\partial R}{\partial s^j_{(1)}}\Delta q^j_{(1)}\Big|_{t_1}^{t_2} \tag{33}$$

将式(32)和式(33)代入式(30),并考虑正则方程(27),经整理后,有

$$\int_{t_1}^{t_2}\{\Delta(R + r^jq^j + u^js^j) + (\frac{\partial R}{\partial s^j}s^j + \frac{\partial R}{\partial s^j_{(1)}})\frac{\mathrm{d}}{\mathrm{d}t}\Delta t + Q^j\Delta q^j - \frac{\partial R}{\partial t}\Delta t\}\mathrm{d}t = (\frac{\partial R}{\partial s^j}\Delta q^j + \frac{\partial R}{\partial s^j_{(1)}}\Delta s^j)\Big|_{t_1}^{t_2} \tag{34}$$

假设

$$\Delta q^j|_{t_1} = \Delta q^j|_{t_2} = 0, \quad \Delta s^j|_{t_1} = \Delta s^j|_{t_2} = 0 \tag{35}$$

于是,式(34)成为

$$\int_{t_1}^{t_2}\{\Delta(R + r^jq^j + u^js^j) + (\frac{\partial R}{\partial s^j}s^j + \frac{\partial R}{\partial s^j_{(1)}}s^j_{(1)})\frac{\mathrm{d}}{\mathrm{d}t}\Delta t + Q^j\Delta q^j - \frac{\partial R}{\partial t}\Delta t\}\mathrm{d}t = 0 \tag{36}$$

方程(36)就是非保守系统(3)的 Raitzin-Tabarrok-Leech 型最小作用量原理.

讨论特殊情况

4.1 对于保守系统,$Q^j = 0$,于是方程(36)成为

$$\int_{t_1}^{t_2}\{\Delta(R + r^jq^j + u^js^j) + (\frac{\partial R}{\partial s^j}s^j + \frac{\partial R}{\partial s^j_{(1)}}s^j_{(1)})\frac{\mathrm{d}}{\mathrm{d}t}\Delta t - \frac{\partial R}{\partial t}\Delta t\}\mathrm{d}t = 0 \tag{37}$$

4.2 若令 $\Delta(r^jq^j + u^js^j) = -(Q^j\Delta q^j - \frac{\partial R}{\partial t}\Delta t)$,对于保守系统,式(36)给出

$$\int_{t_1}^{t_2}\{\Delta R + (\frac{\partial R}{\partial s^j}s^j + \frac{\partial R}{\partial s^j_{(1)}}s^j_{(1)})\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\}\mathrm{d}t = 0 \tag{38}$$

4.3 对于平稳系统,有

$$\frac{\partial R}{\partial s^j}s^j + \frac{\partial R}{\partial s^j_{(1)}}s^j_{(1)} = R \tag{39}$$

于是,在(38)的条件下,式(36)成为

$$\Delta\int_{t_1}^{t_2}R\mathrm{d}x = 0 \tag{40}$$

5 非保守系统的 Lagrange-Raitzin-Tabarrok-Leech 型最小作用量原理

考虑非保守系统(3),在 Raitzin-Tabarrok-Leech 意义下的作用量为

$$G = \int_{t_1}^{t_2}R(t, s^j, s^j_{(1)}, r^j, u^j)\mathrm{d}t \tag{41}$$

于是

$$\Delta G = \int_{t_1}^{t_2}\{\Delta R + R\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\}\mathrm{d}t = \int_{t_1}^{t_2}\delta R\mathrm{d}t + (R\Delta t)\Big|_{t_1}^{t_2} \tag{42}$$

由于

$$\delta R = \frac{\partial R}{\partial s^j}\delta s^j + \frac{\partial R}{\partial s^j_{(1)}}\delta s^j_{(1)} + \frac{\partial R}{\partial r^j}\delta r^j + \frac{\partial R}{\partial u^j}\delta u^j \tag{43}$$

$$\frac{\partial R}{\partial s^j}\delta s^j = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial R}{\partial s^j}\delta q^j) - \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial R}{\partial s^j})\delta q^j,$$

$$\frac{\partial R}{\partial s^j_{(1)}}\delta s^j_{(1)} = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial R}{\partial s^j_{(1)}}\delta s^j) - \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial R}{\partial s^j_{(1)}})\delta s^j \tag{44}$$

将式(44)代入式(43),并注意(26)和(27),得

$$\delta R = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial R}{\partial s^j}\delta q^j) - \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial R}{\partial s^j})\delta q^j + \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial R}{\partial s^j_{(1)}}\delta s^j) - \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial R}{\partial s^j_{(1)}})\delta s^j + \frac{\partial R}{\partial r^j}\delta r^j + \frac{\partial R}{\partial u^j}\delta u^j = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial R}{\partial s^j}\delta q^j) -$$

$$(r^j + Q^j)\delta q^j + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R}{\partial s^j_{(1)}}\delta s^j\right) - q^j\delta r^j - u^j\delta s^j - s^j\delta u^j \tag{45}$$

于是,式(42)可写为

$$\Delta\int_{t_1}^{t_2} R\mathrm{d}t + \int_{t_1}^{t_2}[\delta(r^j q^j + u^j s^j) + Q^j\delta q^j]\mathrm{d}t = [\frac{\partial R}{\partial s^j}\Delta q^j + \frac{\partial R}{\partial s^j_{(1)}}\Delta s^j + R\Delta t]\mid_{t_1}^{t_2} - (\frac{\partial R}{\partial s^j}s^j + \frac{\partial R}{\partial s^j_{(1)}}s^j_{(1)})\Delta t\mid_{t_1}^{t_2} \tag{46}$$

假设

$$\Delta q^j\mid_{t_1} = \Delta q^j\mid_{t_2} = 0,\quad \Delta s^j\mid_{t_1} = \Delta s^j\mid_{t_2} = 0 \tag{47}$$

并注意到,对平稳系统式(39)成立,于是式(46)成为

$$\Delta\int_{t_1}^{t_2} R\mathrm{d}t + \int_{t_1}^{t_2}[\delta(r^j q^j + u^j s^j) + Q^j\delta q^j]\mathrm{d}t = 0 \tag{48}$$

方程(48)就是广义非保守平稳系统的 Lagrange-Raitzin-Tabarrok-Leech 最小作用量原理.

6　举 例

6.1　例1

一棱柱形梁受轴向压力 F 和横向分布载荷 $y(x)=x$,由小挠度理论,已知系统的 Lagrange 函数为

$$L = \frac{EI}{2}q^2_{(2)} - xq - \frac{F}{2}q^2_{(1)} \tag{49}$$

其中 EI 是抗弯刚度,$EI=\text{const}$,试求系统的运动微分方程.

解:本题中将变量 x 视为原理(21)中的时间 t,然后求解,则有

$$v = \frac{\partial L}{\partial q_{(2)}} = EIq_{(2)},\ p = \frac{\partial L}{\partial q_{(1)}} - \frac{\mathrm{d}v}{\mathrm{d}x} = -Fq_{(1)} - EIq_{(3)} \tag{50}$$

$$H = pq_{(1)} + vq_{(2)} - L = pq_{(1)} + xq + \frac{F}{2}q^2_{(1)} + \frac{v^2}{2EI} \tag{51}$$

由于 Hamilton 函数(51)中显含 x,所以系统不存在能量积分,令

$$H = pq_{(1)} + vq_{(2)} - L = h \neq \text{const} \tag{52}$$

非势广义力 $Q=0$,于是原理(21)给出

$$\begin{aligned}\Delta\int_{x_1}^{x_2}(L + h + xh_{(1)})\mathrm{d}x &= \int_{x_1}^{x_2}[\delta L + \delta h + \delta(xh_{(1)})]\mathrm{d}x + [pq_{(1)} + vq_{(2)} + xh_{(1)}]\Delta x\mid_{x_1}^{x_2} \\ &= \int_{x_1}^{x_2}\{\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial q_{(1)}}\delta q_{(1)} + \frac{\partial L}{\partial q_{(2)}}\delta q_{(2)} + \delta h + \frac{\mathrm{d}}{\mathrm{d}x}(x\delta h) - \delta h\}\mathrm{d}x + [pq_{(1)} + vq_{(2)} + xh_{(1)}]\Delta x\mid_{x_1}^{x_2} \\ &= \int_{x_1}^{x_2}\{\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial q_{(1)}} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\frac{\partial L}{\partial q_{(2)}})\}\delta q\mathrm{d}x + [\frac{\partial L}{\partial q_{(1)}}\delta q + \frac{\partial L}{\partial q_{(2)}}\delta q_{(1)} - \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial L}{\partial q_{(2)}})\delta q] + x\delta h + [pq_{(1)} + vq_{(2)} + xh_{(1)}]\Delta x\mid_{x_1}^{x_2} \\ &= \int_{x_1}^{x_2}\{\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial q_{(1)}} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\frac{\partial L}{\partial q_{(2)}})\}\delta q\mathrm{d}x + (p\Delta q + v\Delta q_{(1)} + x\Delta h)x\mid_{x_1}^{x_2}\end{aligned} \tag{53}$$

由于

$$\begin{aligned}&\Delta q\mid_{x_1} = \Delta q\mid_{x_2} = 0,\quad \Delta h\mid_{x_1} = \Delta h\mid_{x_2} = 0 \\ &\Delta s\mid_{x_1} = \Delta s\mid_{x_2} = 0\quad (\Delta s = \Delta q_{(1)})\end{aligned} \tag{54}$$

于是式(53)成为

$$\int_{x_1}^{x_2}\{\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial q_{(1)}} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\frac{\partial L}{\partial q_{(2)}})\}\delta q\mathrm{d}x = 0 \tag{55}$$

因为 δq 是独立的,于是由(55)得

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial q_{(1)}} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial L}{\partial q_{(2)}} = 0 \tag{56}$$

将式(49)代入上式,有

$$-x + \frac{\mathrm{d}}{\mathrm{d}x}(Fq_{(1)}) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}(EIq_{(2)}) = 0 \tag{57}$$

或

$$F\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} + EI\frac{\mathrm{d}^4 q}{\mathrm{d}x^4} = x \tag{58}$$

6.2　例2

力学系统的 Lagrange 函数为

$$L = \frac{1}{2}q^2_{(1)} + \frac{1}{2}q^2_{(2)} \tag{59}$$

非势广义力

$$Q = -q_{(2)} \tag{60}$$

试求系统的运动微分方程.

解:引入 Raitzin 变量及函数

$$\begin{aligned}&s = q_{(1)},\quad s_{(1)} = q_{(2)},\quad r = \frac{\partial L}{\partial q} = 0 \\ &u = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q_{(2)}} = q_{(3)} = s_{(2)} \\ &\tilde{L} = \frac{1}{2}s^2 + \frac{1}{2}s^2_{(1)},\quad Q = -s_{(1)} \\ &R = \tilde{L} = rq - us = \frac{1}{2}s^2 + \frac{1}{2}s^2_{(1)} - us\end{aligned} \tag{61}$$

而

$$\delta R = \frac{\partial R}{\partial s}\delta s + \frac{\partial R}{\partial s_{(1)}}\delta s_{(1)} + \frac{\partial R}{\partial u}\delta u \tag{62}$$

考虑方程(26)和(27),运用分部积分法,原理(48)可写为

$$\int_{t_1}^{t_2}\delta R\mathrm{d}t + (R\Delta t)\mid_{t_1}^{t_2} + \int_{t_1}^{t_2}(q\delta r + r\delta q + u\delta s + s\delta u +$$

$$Q\delta q)\mathrm{d}t = \int_{t_1}^{t_2}(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s} + Q)\delta q\mathrm{d}t + (\frac{\partial R}{\partial s}\Delta q) + \frac{\partial R}{\partial s_{(1)}}\Delta s\mid_{t_1}^{t_2} = 0 \tag{63}$$

假设

$$\Delta q\mid_{t_1} = \Delta q\mid_{t_2} = 0, \quad \Delta s\mid_{t_1} = \Delta s\mid_{t_2} = 0 \tag{64}$$

又 δq 是独立的,于是由方程(63)得

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s} + Q = 0 \tag{65}$$

由于

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s} = s_{(1)} - u_{(1)} = s_{(1)} - s_{(3)}, Q = -s_{(1)} \tag{66}$$

将式(66)代入方程(65),有

$$-2s_{(1)} + s_{(3)} = 0 \tag{67}$$

即

$$2q_{(2)} - q_{(4)} = 0 \tag{68}$$

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PRINCIPLES OF NEW FORM LEAST ACTION OF GENERALIZED NONCONSERVATIVE SYSTEMS *

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Abstract The canonical equations of Hamilton – Tabarrok – Leech and Raitzin – Tabarrok – Leech for generalized nonconservative systems were established respectively. Three kinds of the principles of new form least action for generalized nonconservative systems were given, namely, the principles of new form least action of Lagrange – Tabarrok – Leech, the principles of new form least action of Raitzin – Tabarrok – Leech and the principles of new form least action of Lagrange – Raitzin – Tabarrok – Leech. And two examples were given to illustrate the application of the results.

Key words generalized classical mechanics, nonconservative system, principle of least action

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